CS 1200 Discrete Math Logic of Compound Statements

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- A statement (or proposition) is a sentence that is either true or false. Therefore a sentence, "he is a college student" is not a statement.
- ♦ A compound statement can be made out of simpler statements by using logical operators \land (and), \lor (or), \neg (not).

• If p and q are two statements then the sentence $p \land q$ is called conjunction of p and q and $p \lor q$ is called disjunction of p and q.

♦Note: ¬ has precedence over ∧ and ∨ operators:

Example: $\neg p \land q = (\neg p) \land q$

Sometimes parenthesis are used to define the order of operations:

Example: \neg (p \land q) represents the negation of conjunction of p and q.

A compound statement is a sentence that is either true or false.

♦A truth table is used to represent the relationship of a statement with its negation:

$$\begin{array}{c} P & \neg P \\ T & F \\ F & T \end{array}$$

• If p and q are statement variables, the conjunction of p and q is true only when p and q are true, otherwise it is false

$$\begin{array}{c|cccc} p & q & p \land q \\ \hline T & T & T \\ T & F & F \\ F & T & F \\ F & F & F \\ \hline F & F & F \end{array}$$

• If p and q are statement variables, the disjunction of p and q is false only when p and q are false, otherwise it is true

$$\begin{array}{c|cccc} p & q & p \lor q \\ \hline T & T & T \\ T & F & T \\ F & T & T \\ F & F & F \end{array}$$



р	q	$p \lor q$	$p \wedge q$	$\neg(p \land q)$	$(p \lor q) \land \neg (p \land q)$
Т	Т	Т	Т	F	F
Τ	F	Т	F	Т	Т
F	Т	Т	F	Т	Т
F	F	F	F	Т	F

Logical Equivalence: Two statements are logically equivalent, iff, they have identical truth table.

Example:
$$\neg (\neg p) = p$$

 $P \neg P \neg (\neg p)$
 $T F T$
 $F T F$
 $F T F$
Identical truth values



 \land and \lor operators are commutative $p \land q = q \land p$ $p \lor q = q \lor p$

 \land and \lor operators are associative $p \land q \land r = p \land (q \land r) = (p \land q) \land r$ $p \lor q \lor r = p \lor (q \lor r) = (p \lor q) \lor r$



 \land distributes over \lor $p \land (q \lor r) = (p \land q) \lor (p \land r)$

 \vee distributes over \land $p \lor (q \land r) = (p \lor q) \land (p \lor r)$

 $p \lor \neg p = t$ and $p \land \neg p = c$

 $p \lor p = p$ and $p \land p = p$

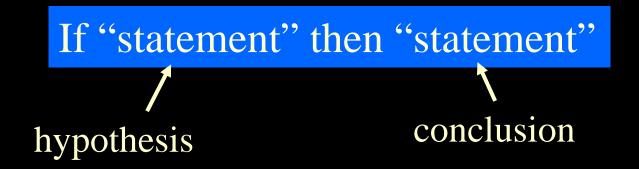
De Morgan's Laws: $\overline{p \land q} = \overline{p} \lor \overline{q}$ $\overline{p \lor q} = \overline{p} \land \overline{q}$

Tautological statements: A tautological statement is a statement that is always true regardless of its components.

Contradictory statements: A contradictory statement is a statement that is always false regardless of its components.

Examples: $p \lor \neg p$ is a tautological statement, and $p \land \neg p$ is a contradictory statement

Conditional Statements: Conditional statements are intended to deduce facts from a condition, if true. It is of the form:



• Conditional Statements: Assume p and q are two sentences, a sentence of the form

If p then q is symbolically written as

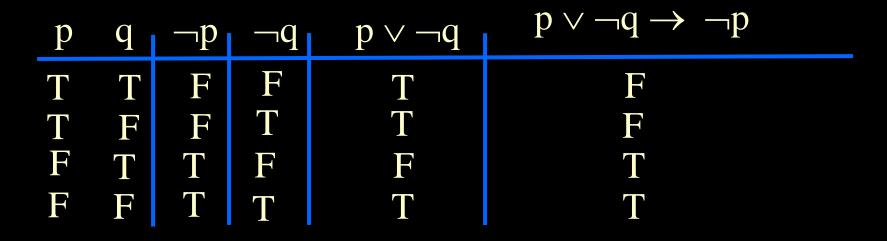


Conditional Statements: Similar to the \neg , \land , \lor operations discussed earlier, we can use the truth table to define $p \rightarrow q$ as a statement.

Conditional Statements: If *p* and *q* are two statement variables, the statement $p \rightarrow q$ is false when *p* is true and *q* is false, otherwise it is true.

$$\begin{array}{c|cccc} p & q & p \rightarrow q \\ \hline T & T & T \\ T & F & F \\ F & T & T \\ F & F & T \\ F & F & T \end{array}$$

 $\bullet \text{Show } p \lor \neg q \to \neg p$



conclusion

hypothesis

Assume p, q, and r are three statements, show the following statements are logically equivalent:

$$p \lor q \rightarrow r \equiv (p \rightarrow r) \land (q \rightarrow r)$$

p	q	r	$p \lor q$	$p \rightarrow r$	$q \rightarrow r$	$p \lor q \rightarrow r$	$(p \rightarrow r) \land (q \rightarrow r)$
Τ	Т	Т	Т	Т	Т	Т	Т
Τ	Т	F	Т	F	F	F	F
Τ	F	Τ	Т	Т	Т	Т	F
Τ	F	F	Т	F	F	F	F
F	Τ	Т	Т	Т	Т	Т	Т
F	Т	F	Т	Т	F	F	F
F	F	Т	F	Т	Т	Т	Т
F	F	F	F	Т	Т	Т	Т



 $(p \rightarrow q) \equiv \neg p \lor q$

р	q	$p \rightarrow q$	_ p	$\neg p \lor q$
Τ	Т	Т	F	Т
Τ	F	F	F	F
F	Т	Т	Т	Т
F	F	Т	Т	Т

Negation of a conditional statement:

 $\neg \ (p \rightarrow q) \equiv p \land \neg q$

As before, the truth table to show the validity of this logical equivalence, or we can use previous relationships to prove its validity:

 $(p \rightarrow q) \equiv \neg p \lor q$ $\neg (p \rightarrow q) \equiv \neg (\neg p \lor q)$ $\equiv \neg (\neg p) \land \neg (q)$ $\equiv (p) \land \neg (q)$

Contrapositive of a conditional statement of the form If p then q is: If ¬q then ¬p

 Corollary: A conditional statement is logically equivalent to its contrapositive

Converse and inverse of a conditional statement are:

If p then q	$p \rightarrow q$	
If q then p	$q \rightarrow p$	converse
If \neg p then \neg q	$\neg p \rightarrow \neg q$	inverse

Only if statement: To say "p only if q" means p can take place only if q takes place.

p only if $q \equiv if \neg q$ then $\neg p \equiv if p$ then q

р	q	if p then q	¬ p	¬ q	if $\neg q$ then $\neg p$
Τ	Т	Т	F	F	Т
Τ	F	F	F	Т	F
F	Т	Т	Т	F	Т
F	F	Т	Т	T	Т

Siconditional statement: assume statement variable p and q, the biconditional of p and q denoted as $p \leftrightarrow q$ is true if both p and q have the same truth values and false otherwise.

p if and only if $q \equiv p$ iff $q \equiv p \leftrightarrow q$

$$\begin{array}{c|cccc} p & q & p \leftrightarrow q \\ T & T & T \\ T & F & F \\ F & T & F \\ F & F & T \end{array}$$

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

р	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \to q) \land (q \to p)$
Τ	Т	Т	Т	Т	Т
Τ	F	F	Т	F	F
F	Т	Т	F	F	F
F	F	Т	Т	Т	Т

Necessary and sufficient conditions: if r and s are statements then:

r is sufficient condition for $s \implies$ if r then s

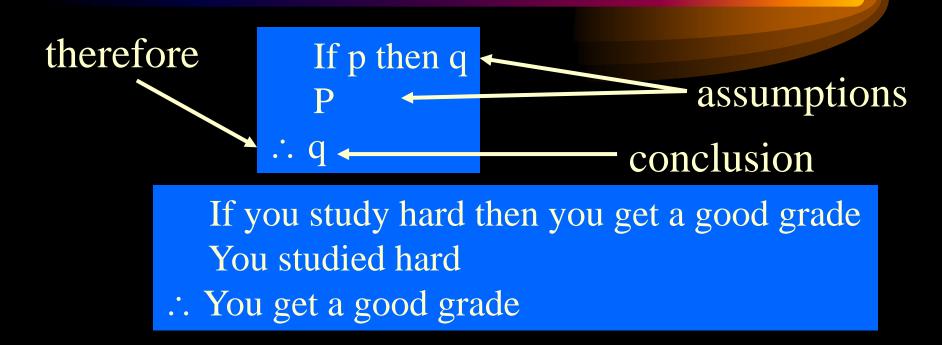
s is a necessary condition for $r \Rightarrow if \neg r$ then $\neg s$

Note, as we discussed before a conditional statement is equivalent to its contrapositive then

if \neg r then \neg s \equiv if s then r \therefore r is a necessary and sufficient condition for s \equiv r iff s

Arguments:

- *An argument is a sequence of statements. An argument form is a sequence of statement forms.
- *All statements or statement forms, except for the last one, is called assumptions or hypotheses. The final statement is called conclusion.





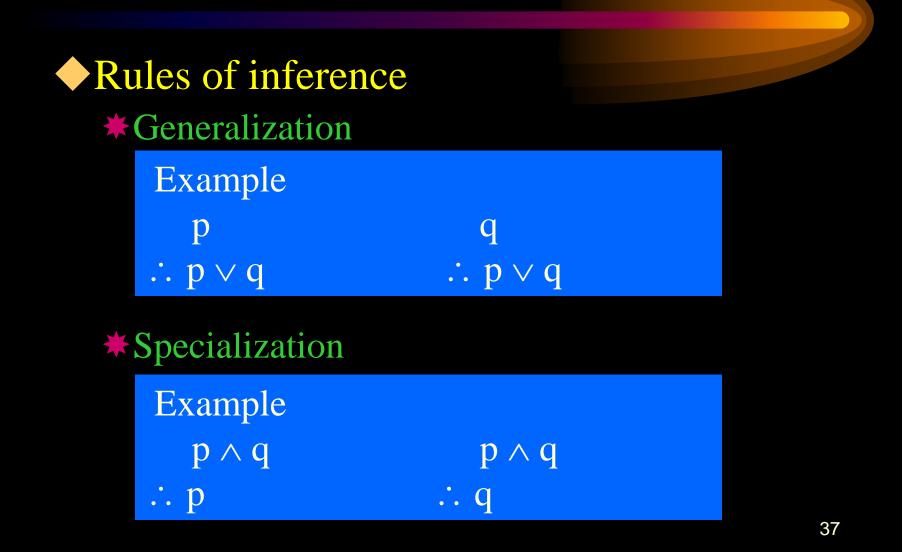
*Note; for a valid argument, the truth of conclusion is deduced from the truth of its hypotheses. Reading between the lines, it is impossible to have a valid argument with true hypotheses and a false conclusion.

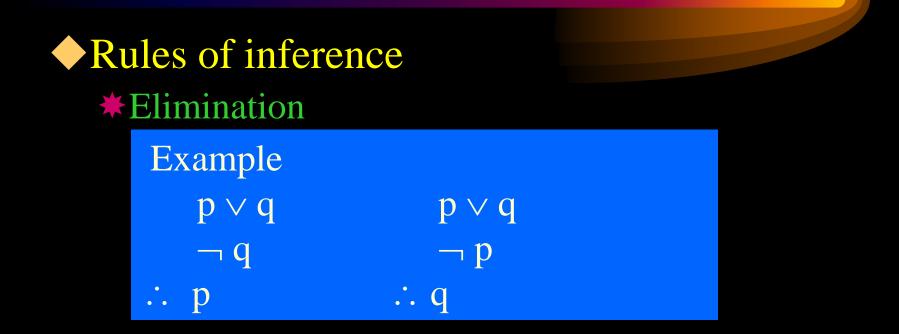
Testing for validity of an argument

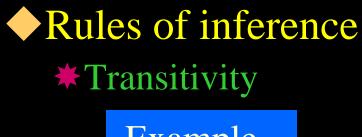
Identify hypotheses and conclusion
Generate its truth table
If the conclusion in every crucial row is true, then the argument form is valid, otherwise, it is invalid.

Example $p \rightarrow q \lor \neg r$ $q \rightarrow p \land r$ $\therefore p \rightarrow r$

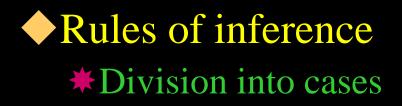
p	q	r	¬r	$q \lor \neg r$	$p \wedge r$	$p \rightarrow q \lor \neg r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
Т	Τ	Τ	F	Т	Т	Т	Т	Т
Τ	Т	F	Т	Т	F	Т	F	
Т	F	Τ	F	F	Т	F	Т	
Τ	F	F	Т	Т	F 🤇	Т	Т	F
F	Τ	Т	F	Т	F	Т	F	
F	Т	F	Т	Т	F	Т	F	
F	F	Τ	F	F	F	Т	Т	Т
F	F	F	Т	Т	F	Т	Т	Т



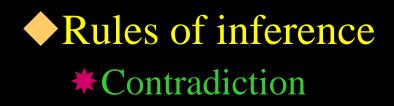




Example $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$



Example $p \lor q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$



If you can show that the supposition that statement *p* is false leads logically to a contradiction, then you can conclude that *p* is true