CS 1200 Discrete Math
Logic of Compound Statements

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Logic of Compound Statements

- A statement (or proposition) is a sentence that is either true or false. Therefore a sentence, “he is a college student” is not a statement.
- A compound statement can be made out of simpler statements by using logical operators $\wedge$ (and), $\vee$ (or), $\neg$ (not).

Logic of Compound Statements

- If *p* and *q* are two statements then the sentence $p \wedge q$ is called conjunction of *p* and *q* and $p \vee q$ is called disjunction of *p* and *q*.

Logic of Compound Statements

- Note: $\neg$ has precedence over $\wedge$ and $\vee$ operators:

  Example: $\neg p \wedge q = (\neg p) \wedge q$

- Sometimes parenthesis are used to define the order of operations:

  Example: $\neg (p \wedge q)$ represents the negation of conjunction of p and q.

Logic of Compound Statements

- A compound statement is a sentence that is either true or false.

Logic of Compound Statements

- A truth table is used to represent the relationship of a statement with its negation:

P	$\neg P$
T	F
F	T

Logic of Compound Statements

- If *p* and *q* are statement variables, the conjunction of *p* and *q* is true only when *p* and *q* are true, otherwise it is false

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logic of Compound Statements

- If *p* and *q* are statement variables, the disjunction of *p* and *q* is false only when *p* and *q* are false, otherwise it is true

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logic of Compound Statements

- Ex-Or

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Logic of Compound Statements

- Logical Equivalence: Two statements are logically equivalent, iff, they have identical truth table.

Example: ¬ (¬ p) = p

P	¬P	¬ (¬ p)
T	F	T
F	T	F

Identical truth values

Logic of Compound Statements

- Some observations:

$\wedge$ and $\vee$ operators are commutative

$$p \wedge q = q \wedge p$$
$$p \vee q = q \vee p$$

$\wedge$ and $\vee$ operators are associative

$$p \wedge q \wedge r = p \wedge (q \wedge r) = (p \wedge q) \wedge r$$
$$p \vee q \vee r = p \vee (q \vee r) = (p \vee q) \vee r$$

Logic of Compound Statements

- Some observations:

$\wedge$ distributes over $\vee$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$\vee$ distributes over $\wedge$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$p \vee \neg p = t$ and $p \wedge \neg p = c$

$p \vee p = p$ and $p \wedge p = p$

Logic of Compound Statements

- De Morgan’s Laws:

$$\overline{p \wedge q} = \overline{p} \vee \overline{q}$$

$$\overline{p \vee q} = \overline{p} \wedge \overline{q}$$

Logic of Compound Statements

- Tautological statements: A tautological statement is a statement that is always true regardless of its components.

- Contradictory statements: A contradictory statement is a statement that is always false regardless of its components.

Logic of Compound Statements

Examples:

$p \vee \neg p$ is a tautological statement, and

$p \wedge \neg p$ is a contradictory statement

Logic of Compound Statements

- Conditional Statements: Conditional statements are intended to deduce facts from a condition, if true. It is of the form:

If "statement" then "statement"

hypothesis

conclusion

Logic of Compound Statements

- Conditional Statements: Assume *p* and *q* are two sentences, a sentence of the form

If p then q

is symbolically written as

$p \to q$

Logic of Compound Statements

- Conditional Statements: Similar to the $\neg$, $\wedge$, $\vee$ operations discussed earlier, we can use the truth table to define $p \rightarrow q$ as a statement.

Logic of Compound Statements

- Conditional Statements: If p and q are two statement variables, the statement $p \to q$ is false when p is true and q is false, otherwise it is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Logic of Compound Statements

- Show $p \vee \neg q \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$p \vee \neg q \rightarrow \neg p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

conclusion hypothesis

Logic of Compound Statements

- Assume *p*, *q*, and *r* are three statements, show the following statements are logically equivalent:

$$p \vee q \to r \equiv (p \to r) \wedge (q \to r)$$

Logic of Compound Statements

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	F
T	F	F	T	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Logic of Compound Statements

- Show

$$(p \rightarrow q) \equiv \neg p \vee q$$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logic of Compound Statements

- Negation of a conditional statement:

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

- As before, the truth table to show the validity of this logical equivalence, or we can use previous relationships to prove its validity:

Logic of Compound Statements

$$(p \rightarrow q) \equiv \neg p \vee q$$

$$\neg (p \rightarrow q) \equiv \neg (\neg p \vee q)$$

$$\equiv \neg (\neg p) \wedge \neg (q)$$

$$\equiv (p) \wedge \neg (q)$$

Logic of Compound Statements

- Contrapositive of a conditional statement of the form If p then q is:
  If ¬ q then ¬ p

- Corollary: A conditional statement is logically equivalent to its contrapositive

Logic of Compound Statements

- Converse and inverse of a conditional statement are:

If p then q	$p \rightarrow q$	
If q then p	$q \rightarrow p$	converse
If ¬ p then ¬ q	$\neg p \rightarrow \neg q$	inverse

Logic of Compound Statements

- Only if statement: To say "p only if q" means *p* can take place only if *q* takes place.

p only if q ≡ if ¬ q then ¬ p ≡ if p then q

p	q	if p then q	¬ p	¬ q	if ¬ q then ¬ p
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Logic of Compound Statements

- Biconditional statement: assume statement variable *p* and *q*, the biconditional of *p* and *q* denoted as p ↔ q is true if both *p* and *q* have the same truth values and false otherwise.

p if and only if q ≡ p iff q ≡ p ↔ q

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Logic of Compound Statements

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Logic of Compound Statements

- Necessary and sufficient conditions: if *r* and s are statements then:

r is sufficient condition for s $\Rightarrow$ if r then s

s is a necessary condition for r $\Rightarrow$ if $\neg$ r then $\neg$ s

- Note, as we discussed before a conditional statement is equivalent to its contrapositive then

if $\neg$ r then $\neg$ s $\equiv$ if s then r

$\therefore$ r is a necessary and sufficient condition for s $\equiv$ r iff s

Logic of Compound Statements

- Arguments:
  - An argument is a sequence of statements. An argument form is a sequence of statement forms.
  - All statements or statement forms, except for the last one, is called assumptions or hypotheses. The final statement is called conclusion.

Logic of Compound Statements

therefore

If p then q
p
∴ q

assumptions

conclusion

If you study hard then you get a good grade
You studied hard
∴ You get a good grade

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- Argument
  - Note; for a valid argument, the truth of conclusion is deduced from the truth of its hypotheses. Reading between the lines, it is impossible to have a valid argument with true hypotheses and a false conclusion.

Logic of Compound Statements

Testing for validity of an argument

1) Identify hypotheses and conclusion
2) Generate its truth table
3) If the conclusion in every crucial row is true, then the argument form is valid, otherwise, it is invalid.

Example

$p \rightarrow q \vee \neg r$

$q \rightarrow p \wedge r$

$\therefore\ p \rightarrow r$

Logic of Compound Statements

p	q	r	¬r	q ∨ ¬ r	p ∧ r	p → q ∨ ¬ r	q → p ∧ r	p → r
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Logic of Compound Statements

- Rules of inference
  - Generalization

Example	
p	q
$\therefore\ p \vee q$	$\therefore\ p \vee q$

  - Specialization

Example	
$p \wedge q$	$p \wedge q$
$\therefore\ p$	$\therefore\ q$

Logic of Compound Statements

- Rules of inference
  - Elimination

Example

$p \vee q$	$p \vee q$
$\neg q$	$\neg p$
$\therefore\ p$	$\therefore\ q$

Logic of Compound Statements

- Rules of inference
  - Transitivity

Example

$$\begin{array}{l} \quad p \rightarrow q \\ \quad q \rightarrow r \\ \therefore\ p \rightarrow r \end{array}$$

Logic of Compound Statements

- Rules of inference
  - Division into cases

Example

$$p \vee q$$
$$p \to r$$
$$q \to r$$
$$\therefore\ r$$

Logic of Compound Statements

- Rules of inference
  - Contradiction

If you can show that the supposition that statement *p* is false leads logically to a contradiction, then you can conclude that *p* is true